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Some fixed point theorems for rational Geraghty contractive mappings in ordered b -metric spaces

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Abstract

In this paper, new classes of rational Geraghty contractive mappings in the setup of b -metric spaces are introduced. Moreover, the existence of some fixed point for such mappings in ordered b -metric spaces are investigated. Also, some examples are provided to illustrate the results presented herein. Finally, an application of the main result is given.

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1 Introduction

Using different forms of contractive conditions in various generalized metric spaces, there is a large number of extensions of the Banach contraction principle [1]. Some of such generalizations are obtained via rational contractive conditions. Recently, Azam *et al.* [2] established some fixed point results for a pair of rational contractive mappings in complex valued metric spaces. Also, in [3], Nashine *et al.* proved some common fixed point theorems for a pair of mappings satisfying certain rational contractions in the framework of complex valued metric spaces. In [4], the authors proved some unique fixed point results for an operator T satisfying certain rational contractive condition in a partially ordered metric space. In fact, their results generalize the main result of Jaggi [5].

Ran and Reurings started the studying of fixed point results on partially ordered sets in [6], where they gave many useful results in matrix equations. Recently, many researchers have focused on different contractive conditions in complete metric spaces endowed with a partial order and obtained many fixed point results in such spaces. For more details on fixed point results in ordered metric spaces we refer the reader to [7, 8] and [9].

Czerwik in [10] introduced the concept of a b -metric space. Since then, several papers dealt with fixed point theory for single-valued and multi-valued operators in b -metric spaces (see, *e.g.*, [11–16] and [17, 18]).

Definition 1 Let X be a (nonempty) set and $s \geq 1$ be a given real number. A function $d : X \times X \rightarrow \mathbb{R}^+$ is a b -metric if the following conditions are satisfied:

$$(b_1) \quad d(x, y) = 0 \text{ iff } x = y,$$

$$\begin{aligned}(b_2) \quad & d(x, y) = d(y, x), \\(b_3) \quad & d(x, z) \leq s[d(x, y) + d(y, z)]\end{aligned}$$

for all $x, y, z \in X$.

In this case, the pair (X, d) is called a b -metric space.

Definition 2 [19] Let (X, d) be a b -metric space.

- (a) A sequence $\{x_n\}$ in X is called b -convergent if and only if there exists $x \in X$ such that $d(x_n, x) \rightarrow 0$ as $n \rightarrow \infty$.
- (b) $\{x_n\}$ in X is said to be b -Cauchy if and only if $d(x_n, x_m) \rightarrow 0$, as $n, m \rightarrow \infty$.
- (c) The b -metric space (X, d) is called b -complete if every b -Cauchy sequence in X is b -convergent.

The following example (corrected from [20]) illustrates that a b -metric need not be a continuous function.

Example 1 Let $X = \mathbb{N} \cup \{\infty\}$ and $d : X \times X \rightarrow \mathbb{R}$ be defined by

$$d(m, n) = \begin{cases} 0, & \text{if } m = n, \\ |\frac{1}{m} - \frac{1}{n}|, & \text{if one of } m, n \text{ is even and the other is even or } \infty, \\ 5, & \text{if one of } m, n \text{ is odd and the other is odd (and } m \neq n) \text{ or } \infty, \\ 2, & \text{otherwise.} \end{cases}$$

Then $d(m, p) \leq \frac{5}{2}(d(m, n) + d(n, p))$ for all $m, n, p \in X$. Thus, (X, d) is a b -metric space (with $s = 5/2$). Let $x_n = 2n$ for each $n \in \mathbb{N}$. So $d(2n, \infty) = \frac{1}{2n} \rightarrow 0$ as $n \rightarrow \infty$ that is, $x_n \rightarrow \infty$, but $d(x_n, 1) = 2 \not\rightarrow 5 = d(\infty, 1)$ as $n \rightarrow \infty$.

Lemma 1 [21] Let (X, d) be a b -metric space with $s \geq 1$, and suppose that $\{x_n\}$ and $\{y_n\}$ are b -convergent to x and y , respectively. Then

$$\frac{1}{s^2}d(x, y) \leq \liminf_{n \rightarrow \infty} d(x_n, y_n) \leq \limsup_{n \rightarrow \infty} d(x_n, y_n) \leq s^2 d(x, y).$$

Moreover, for each $z \in X$, we have

$$\frac{1}{s}d(x, z) \leq \liminf_{n \rightarrow \infty} d(x_n, z) \leq \limsup_{n \rightarrow \infty} d(x_n, z) \leq s d(x, z).$$

Let \mathfrak{S} denote the class of all real functions $\beta : [0, +\infty) \rightarrow [0, 1)$ satisfying the condition

$$\beta(t_n) \rightarrow 1 \quad \text{implies that} \quad t_n \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

In order to generalize the Banach contraction principle, Geraghty proved the following.

Theorem 1 [22] Let (X, d) be a complete metric space, and let $f : X \rightarrow X$ be a self-map. Suppose that there exists $\beta \in \mathfrak{S}$ such that

$$d(fx, fy) \leq \beta(d(x, y))d(x, y)$$

holds for all $x, y \in X$. Then f has a unique fixed point $z \in X$ and for each $x \in X$ the Picard sequence $\{f^n x\}$ converges to z .

Amini-Harandi and Emami [23] generalized the result of Geraghty to the framework of a partially ordered complete metric space as follows.

Theorem 2 Let (X, d, \preceq) be a complete partially ordered metric space. Let $f : X \rightarrow X$ be an increasing self-map such that there exists $x_0 \in X$ with $x_0 \preceq fx_0$. Suppose that there exists $\beta \in \mathfrak{S}$ such that

$$d(fx, fy) \leq \beta(d(x, y))d(x, y)$$

holds for all $x, y \in X$ with $y \preceq x$. Assume that either f is continuous or X is such that if an increasing sequence $\{x_n\}$ in X converges to $x \in X$, then $x_n \preceq x$ for all n . Then f has a fixed point in X . Moreover, if for each $x, y \in X$ there exists $z \in X$ comparable with x and y , then the fixed point of f is unique.

In [24], some fixed point theorems for mappings satisfying Geraghty-type contractive conditions are proved in various generalized metric spaces. As in [24], we will consider the class \mathcal{F} of functions $\beta : [0, \infty) \rightarrow [0, 1/s)$ such that

$$\beta(t_n) \rightarrow \frac{1}{s} \quad \text{implies that} \quad t_n \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

Theorem 3 [24] Let $s > 1$, and let (X, D, s) be a complete metric type space. Suppose that a mapping $f : X \rightarrow X$ satisfies the condition

$$D(fx, fy) \leq \beta(D(x, y))D(x, y)$$

for all $x, y \in X$ and some $\beta \in \mathcal{F}$. Then f has a unique fixed point $z \in X$, and for each $x \in X$ the Picard sequence $\{f^n x\}$ converges to z in (X, D, s) .

Also, by unification of the recent results obtained by Zabihi and Razani [25] we have the following result.

Theorem 4 Let (X, \preceq) be a partially ordered set and suppose that there exists a b -metric d on X such that (X, d) is a b -complete b -metric space (with parameter $s > 1$). Let $f : X \rightarrow X$ be an increasing mapping with respect to \preceq such that there exists an element $x_0 \in X$ with $x_0 \preceq f(x_0)$. Suppose there exists $\beta \in \mathcal{F}$ such that

$$sd(fx, fy) \leq \beta(d(x, y))M(x, y) + LN(x, y) \tag{1.1}$$

for all comparable elements $x, y \in X$, where $L \geq 0$,

$$M(x, y) = \max \left\{ d(x, y), \frac{d(x, fx)d(y, fy)}{1 + d(fx, fy)} \right\}$$

and

$$N(x, y) = \min \{ d(x, fx), d(x, fy), d(y, fx), d(y, fy) \}.$$

If f is continuous, or, whenever $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \rightarrow u \in X$, one has $x_n \leq u$ for all $n \in \mathbb{N}$, then f has a fixed point. Moreover, the set of fixed points of f is well ordered if and only if f has one and only one fixed point.

The aim of this paper is to present some fixed point theorems for rational Geraghty contractive mappings in partially ordered b -metric spaces. Our results extend some existing results in the literature.

2 Main results

Let \mathcal{F} denotes the class of all functions $\beta : [0, \infty) \rightarrow [0, \frac{1}{s})$ satisfying the following condition:

$$\limsup_{n \rightarrow \infty} \beta(t_n) = \frac{1}{s} \quad \text{implies that} \quad t_n \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

Definition 3 Let (X, d, \preceq) be a b -metric space. A mapping $f : X \rightarrow X$ is called a rational Geraghty contraction of type I if there exists $\beta \in \mathcal{F}$ such that

$$d(fx, fy) \leq \beta(M(x, y))M(x, y) \quad (2.1)$$

for all comparable elements $x, y \in X$, where

$$M(x, y) = \max \left\{ d(x, y), \frac{d(x, fx)d(y, fy)}{1 + d(x, y)}, \frac{d(x, fx)d(y, fy)}{1 + d(fx, fy)} \right\}.$$

Theorem 5 Let (X, \preceq) be a partially ordered set and suppose there exists a b -metric d on X such that (X, d) is a b -complete b -metric space (with parameter $s > 1$). Let $f : X \rightarrow X$ be an increasing mapping with respect to \preceq such that there exists an element $x_0 \in X$ with $x_0 \preceq f(x_0)$. Suppose f is a rational Geraghty contraction of type I. If

- (I) f is continuous, or,
- (II) whenever $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \rightarrow u \in X$, one has $x_n \leq u$ for all $n \in \mathbb{N}$,

then f has a fixed point.

Moreover, the set of fixed points of f is well ordered if and only if f has one and only one fixed point.

Proof Let $x_n = f^n(x_0)$ for all $n \geq 0$. Since $x_0 \preceq f(x_0)$ and f is increasing, we obtain by induction that

$$x_0 \preceq f(x_0) \preceq f^2(x_0) \preceq \cdots \preceq f^n(x_0) \preceq f^{n+1}(x_0) \preceq \cdots.$$

We do the proof in the following steps.

Step I: We show that $\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0$. Since $x_n \preceq x_{n+1}$ for each $n \in \mathbb{N}$, then by (2.1)

$$\begin{aligned} d(x_n, x_{n+1}) &= d(fx_{n-1}, fx_n) \\ &\leq \beta(M(x_{n-1}, x_n))M(x_{n-1}, x_n), \end{aligned} \quad (2.2)$$

where

$$\begin{aligned} M(x_{n-1}, x_n) &= \max \left\{ d(x_{n-1}, x_n), \frac{d(x_{n-1}, fx_{n-1})d(x_n, fx_n)}{1 + d(x_{n-1}, x_n)}, \right. \\ &\quad \left. \frac{d(x_{n-1}, fx_{n-1})d(x_n, fx_n)}{1 + d(fx_{n-1}, fx_n)} \right\} \\ &= \max \left\{ d(x_{n-1}, x_n), \frac{d(x_{n-1}, x_n)d(x_n, x_{n+1})}{1 + d(x_{n-1}, x_n)}, \frac{d(x_{n-1}, x_n)d(x_n, x_{n+1})}{1 + d(x_n, x_{n+1})} \right\} \\ &\leq \max \{ d(x_{n-1}, x_n), d(x_n, x_{n+1}) \}. \end{aligned}$$

If $\max \{ d(x_{n-1}, x_n), d(x_n, x_{n+1}) \} = d(x_n, x_{n+1})$, then from (2.2),

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \beta(M(x_n, x_{n+1}))d(x_n, x_{n+1}) \\ &< \frac{1}{s}d(x_n, x_{n+1}) \\ &< d(x_n, x_{n+1}), \end{aligned} \tag{2.3}$$

which is a contradiction.

Hence, $\max \{ d(x_{n-1}, x_n), d(x_n, x_{n+1}) \} = d(x_{n-1}, x_n)$, so from (2.2),

$$d(x_n, x_{n+1}) \leq \beta(M(x_{n-1}, x_n))d(x_{n-1}, x_n). \tag{2.4}$$

Since $\{d(x_n, x_{n+1})\}$ is a decreasing sequence, then there exists $r \geq 0$ such that $\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = r$. We prove $r = 0$. Suppose on contrary that $r > 0$. Then, letting $n \rightarrow \infty$, from (2.4) we have

$$r \leq \lim_{n \rightarrow \infty} \beta(M(x_{n-1}, x_n))r,$$

which implies that $\frac{1}{s} \leq 1 \leq \lim_{n \rightarrow \infty} \beta(M(x_{n-1}, x_n))$. Now, as $\beta \in \mathcal{F}$ we conclude that $M(x_{n-1}, x_n) \rightarrow 0$, which yields $r = 0$, a contradiction. Hence, $r = 0$. That is,

$$\lim_{n \rightarrow \infty} d(x_{n-1}, x_n) = 0. \tag{2.5}$$

Step II: Now, we prove that the sequence $\{x_n\}$ is a b -Cauchy sequence. Suppose the contrary, *i.e.*, $\{x_n\}$ is not a b -Cauchy sequence. Then there exists $\varepsilon > 0$ for which we can find two subsequences $\{x_{m_i}\}$ and $\{x_{n_i}\}$ of $\{x_n\}$ such that n_i is the smallest index for which

$$n_i > m_i > i \quad \text{and} \quad d(x_{m_i}, x_{n_i}) \geq \varepsilon. \tag{2.6}$$

This means that

$$d(x_{m_i}, x_{n_i-1}) < \varepsilon. \tag{2.7}$$

From (2.5) and using the triangular inequality, we get

$$\varepsilon \leq d(x_{m_i}, x_{n_i}) \leq sd(x_{m_i}, x_{m_i+1}) + sd(x_{m_i+1}, x_{n_i}).$$

By taking the upper limit as $i \rightarrow \infty$, we get

$$\frac{\varepsilon}{s} \leq \limsup_{i \rightarrow \infty} d(x_{m_i+1}, x_{n_i}). \quad (2.8)$$

The definition of $M(x, y)$ and (2.8) imply

$$\begin{aligned} & \limsup_{i \rightarrow \infty} M(x_{m_i}, x_{n_i-1}) \\ &= \limsup_{i \rightarrow \infty} \max \left\{ d(x_{m_i}, x_{n_i-1}), \frac{d(x_{m_i}, fx_{m_i})d(x_{n_i-1}, fx_{n_i-1})}{1 + d(x_{m_i}, x_{n_i-1})}, \right. \\ & \quad \left. \frac{d(x_{m_i}, fx_{m_i})d(x_{n_i-1}, fx_{n_i-1})}{1 + d(fx_{m_i}, fx_{n_i-1})} \right\} \\ &= \limsup_{i \rightarrow \infty} \max \left\{ d(x_{m_i}, x_{n_i-1}), \frac{d(x_{m_i}, x_{m_i+1})d(x_{n_i-1}, x_{n_i})}{1 + d(x_{m_i}, x_{n_i-1})}, \right. \\ & \quad \left. \frac{d(x_{m_i}, x_{m_i+1})d(x_{n_i-1}, x_{n_i})}{1 + d(x_{m_i+1}, x_{n_i})} \right\} \\ &\leq \varepsilon. \end{aligned}$$

Now, from (2.1) and the above inequalities, we have

$$\begin{aligned} \frac{\varepsilon}{s} &\leq \limsup_{i \rightarrow \infty} d(x_{m_i+1}, x_{n_i}) \\ &\leq \limsup_{i \rightarrow \infty} \beta(M(x_{m_i}, x_{n_i-1})) \limsup_{i \rightarrow \infty} M(x_{m_i}, x_{n_i-1}) \\ &\leq \varepsilon \limsup_{i \rightarrow \infty} \beta(M(x_{m_i}, x_{n_i-1})), \end{aligned}$$

which implies that $\frac{1}{s} \leq \limsup_{i \rightarrow \infty} \beta(M(x_{m_i}, x_{n_i-1}))$. Now, as $\beta \in \mathcal{F}$ we conclude that $M(x_{m_i}, x_{n_i-1}) \rightarrow 0$, which yields $d(x_{m_i}, x_{n_i-1}) \rightarrow 0$. Consequently,

$$d(x_{m_i}, x_{n_i}) \leq sd(x_{m_i}, x_{n_i-1}) + sd(x_{n_i-1}, x_{n_i}) \rightarrow 0,$$

which is a contradiction to (2.6). Therefore, $\{x_n\}$ is a b -Cauchy sequence. b -Completeness of X shows that $\{x_n\}$ b -converges to a point $u \in X$.

Step III: u is a fixed point of f .

First, let f be continuous, so we have

$$u = \lim_{n \rightarrow \infty} x_{n+1} = \lim_{n \rightarrow \infty} fx_n = fu.$$

Now, let (II) holds. Using the assumption on X we have $x_n \leq u$. Now, we show that $u = fu$. By Lemma 1

$$\begin{aligned} \frac{1}{s} d(u, fu) &\leq \limsup_{n \rightarrow \infty} d(x_{n+1}, fu) \\ &\leq \limsup_{n \rightarrow \infty} \beta(M(x_n, u)) \limsup_{n \rightarrow \infty} M(x_n, u), \end{aligned}$$

where

$$\begin{aligned}\lim_{n \rightarrow \infty} M(x_n, u) &= \lim_{n \rightarrow \infty} \max \left\{ d(x_n, u), \frac{d(x_n, fx_n)d(u, fu)}{1 + d(x_n, u)}, \frac{d(x_n, fx_n)d(u, fu)}{1 + d(fx_n, fu)} \right\} \\ &= \max\{0, 0\} \\ &= 0.\end{aligned}$$

Therefore, from the above relations, we deduce that $d(u, fu) = 0$, so $u = fu$.

Finally, suppose that the set of fixed point of f is well ordered. Assume to the contrary that u and v are two fixed points of f such that $u \neq v$. Then by (2.1),

$$d(u, v) = d(fu, fv) \leq \beta(M(u, v))M(u, v) = \beta(d(u, v))d(u, v) < \frac{1}{s}d(u, v), \quad (2.9)$$

because

$$M(u, v) = \max \left\{ d(u, v), \frac{d(u, u)d(v, v)}{1 + d(u, v)} \right\} = d(u, v).$$

So we get $d(u, v) < \frac{1}{s}d(u, v)$, a contradiction. Hence $u = v$, and f has a unique fixed point. Conversely, if f has a unique fixed point, then the set of fixed points of f is a singleton, and so it is well ordered. \square

Definition 4 Let (X, d) be a b -metric space. A mapping $f : X \rightarrow X$ is called a rational Geraghty contraction of type II if there exists $\beta \in \mathcal{F}$ such that

$$d(fx, fy) \leq \beta(M(x, y))M(x, y) \quad (2.10)$$

for all comparable elements $x, y \in X$, where

$$\begin{aligned}M(x, y) &= \max \left\{ d(x, y), \frac{d(x, fx)d(x, fy) + d(y, fy)d(y, fx)}{1 + s[d(x, fx) + d(y, fy)]}, \right. \\ &\quad \left. \frac{d(x, fx)d(x, fy) + d(y, fy)d(y, fx)}{1 + d(x, fy) + d(y, fx)} \right\}.\end{aligned}$$

Theorem 6 Let (X, \preceq) be a partially ordered set and suppose that there exists a b -metric d on X such that (X, d) is a b -complete b -metric space. Let $f : X \rightarrow X$ be an increasing mapping with respect to \preceq such that there exists an element $x_0 \in X$ with $x_0 \preceq f(x_0)$. Suppose f is a rational Geraghty contractive mapping of type II. If

- (I) f is continuous, or,
- (II) whenever $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \rightarrow u \in X$, one has $x_n \preceq u$ for all $n \in \mathbb{N}$,

then f has a fixed point.

Moreover, the set of fixed points of f is well ordered if and only if f has one and only one fixed point.

Proof Set $x_n = f^n(x_0)$. Since $x_0 \preceq f(x_0)$ and f is increasing, we obtain by induction that

$$x_0 \preceq f(x_0) \preceq f^2(x_0) \preceq \cdots \preceq f^n(x_0) \preceq f^{n+1}(x_0) \preceq \cdots.$$

We do the proof in the following steps.

Step I: We show that $\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0$. Since $x_n \preceq x_{n+1}$ for each $n \in \mathbb{N}$, then by (2.10)

$$\begin{aligned} d(x_n, x_{n+1}) &= d(fx_{n-1}, fx_n) \\ &\leq \beta(M(x_{n-1}, x_n))M(x_{n-1}, x_n) \\ &\leq \beta(d(x_{n-1}, x_n))d(x_{n-1}, x_n) \\ &< \frac{1}{s}d(x_{n-1}, x_n) \\ &\leq d(x_{n-1}, x_n), \end{aligned} \quad (2.11)$$

because

$$\begin{aligned} M(x_{n-1}, x_n) &= \max \left\{ d(x_{n-1}, x_n), \frac{d(x_{n-1}, fx_{n-1})d(x_{n-1}, fx_n) + d(x_n, fx_n)d(x_n, fx_{n-1})}{1 + s[d(x_{n-1}, fx_{n-1}) + d(x_n, fx_n)]}, \right. \\ &\quad \left. \frac{d(x_{n-1}, fx_{n-1})d(x_{n-1}, fx_n) + d(x_n, fx_n)d(x_n, fx_{n-1})}{1 + d(x_{n-1}, fx_n) + d(x_n, fx_{n-1})} \right\} \\ &= \max \left\{ d(x_{n-1}, x_n), \frac{d(x_{n-1}, x_n)d(x_{n-1}, x_{n+1}) + d(x_n, x_{n+1})d(x_n, x_n)}{1 + s[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]}, \right. \\ &\quad \left. \frac{d(x_{n-1}, x_n)d(x_{n-1}, x_{n+1}) + d(x_n, x_{n+1})d(x_n, x_n)}{1 + d(x_{n-1}, x_{n+1}) + d(x_n, x_n)} \right\} \\ &= d(x_{n-1}, x_n). \end{aligned}$$

Therefore, $\{d(x_n, x_{n+1})\}$ is decreasing. Then there exists $r \geq 0$ such that $\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = r$. We will prove that $r = 0$. Suppose to the contrary that $r > 0$. Then, letting $n \rightarrow \infty$, from (2.11)

$$\frac{1}{s}r \leq \lim_{n \rightarrow \infty} \beta(d(x_{n-1}, x_n))r,$$

which implies that $d(x_{n-1}, x_n) \rightarrow 0$. Hence, $r = 0$, a contradiction. So,

$$\lim_{n \rightarrow \infty} d(x_{n-1}, x_n) = 0 \quad (2.12)$$

holds true.

Step II: Now, we prove that the sequence $\{x_n\}$ is a b -Cauchy sequence. Suppose the contrary, i.e., $\{x_n\}$ is not a b -Cauchy sequence. Then there exists $\varepsilon > 0$ for which we can find two subsequences $\{x_{m_i}\}$ and $\{x_{n_i}\}$ of $\{x_n\}$ such that n_i is the smallest index for which

$$n_i > m_i > i \quad \text{and} \quad d(x_{m_i}, x_{n_i}) \geq \varepsilon. \quad (2.13)$$

This means that

$$d(x_{m_i}, x_{n_i-1}) < \varepsilon. \quad (2.14)$$

As in the proof of Theorem 5, we have

$$\frac{\varepsilon}{s} \leq \limsup_{i \rightarrow \infty} d(x_{m_i+1}, x_{n_i}). \quad (2.15)$$

From the definition of $M(x, y)$ and the above limits,

$$\begin{aligned} & \limsup_{i \rightarrow \infty} M(x_{m_i}, x_{n_i-1}) \\ &= \limsup_{i \rightarrow \infty} \max \left\{ d(x_{m_i}, x_{n_i-1}), \right. \\ & \quad \frac{d(x_{m_i}, fx_{m_i})d(x_{m_i}, fx_{n_i-1}) + d(x_{n_i-1}, fx_{n_i-1})d(x_{n_i-1}, fx_{m_i})}{1 + s[d(x_{m_i}, fx_{m_i}) + d(x_{n_i-1}, fx_{n_i-1})]}, \\ & \quad \left. \frac{d(x_{m_i}, fx_{m_i})d(x_{m_i}, fx_{n_i-1}) + d(x_{n_i-1}, fx_{n_i-1})d(x_{n_i-1}, fx_{m_i})}{1 + d(x_{m_i}, fx_{n_i-1}) + d(x_{n_i-1}, fx_{m_i})} \right\} \\ &= \limsup_{i \rightarrow \infty} \max \left\{ d(x_{m_i}, x_{n_i-1}), \right. \\ & \quad \frac{d(x_{m_i}, x_{m_i+1})d(x_{m_i}, x_{n_i}) + d(x_{n_i-1}, x_{n_i})d(x_{n_i-1}, x_{m_i+1})}{1 + s[d(x_{m_i}, x_{m_i+1}) + d(x_{n_i-1}, x_{n_i})]}, \\ & \quad \left. \frac{d(x_{m_i}, x_{m_i+1})d(x_{m_i}, x_{n_i}) + d(x_{n_i-1}, x_{n_i})d(x_{n_i-1}, x_{m_i+1})}{1 + d(x_{m_i}, x_{n_i}) + d(x_{n_i-1}, x_{m_i+1})} \right\} \\ &\leq \varepsilon. \end{aligned}$$

Now, from (2.10) and the above inequalities, we have

$$\begin{aligned} \frac{\varepsilon}{s} &\leq \limsup_{i \rightarrow \infty} d(x_{m_i+1}, x_{n_i}) \leq \limsup_{i \rightarrow \infty} \beta(M(x_{m_i}, x_{n_i-1})) \limsup_{i \rightarrow \infty} M(x_{m_i}, x_{n_i-1}) \\ &\leq \varepsilon \limsup_{i \rightarrow \infty} \beta(M(x_{m_i}, x_{n_i-1})), \end{aligned}$$

which implies that $\frac{1}{s} \leq \limsup_{i \rightarrow \infty} \beta(M(x_{m_i}, x_{n_i-1}))$. Now, as $\beta \in \mathcal{F}$ we conclude that $\{x_n\}$ is a b -Cauchy sequence. b -Completeness of X shows that $\{x_n\}$ b -converges to a point $u \in X$.

Step III: u is a fixed point of f .

First, let f be continuous, so we have

$$u = \lim_{n \rightarrow \infty} x_{n+1} = \lim_{n \rightarrow \infty} fx_n = fu.$$

Now, let (II) hold. Using the assumption on X we have $x_n \leq u$. Now, we show that $u = fu$. By Lemma 1

$$\begin{aligned} \frac{1}{s} d(u, fu) &\leq \limsup_{n \rightarrow \infty} d(x_{n+1}, fu) \\ &\leq \limsup_{n \rightarrow \infty} \beta(M(x_n, u)) \limsup_{n \rightarrow \infty} M(x_n, u) \\ &= 0, \end{aligned}$$

because

$$\begin{aligned} \lim_{n \rightarrow \infty} M(x_n, u) &= \lim_{n \rightarrow \infty} \max \left\{ d(x_n, u), \frac{d(x_n, fx_n)d(x_n, fu) + d(u, fu)d(u, fx_n)}{1 + s[d(x_n, fx_n) + d(u, fu)]}, \right. \\ & \quad \left. \frac{d(x_n, fx_n)d(x_n, fu) + d(u, fu)d(u, fx_n)}{1 + d(x_n, fu) + d(x_n, fu)} \right\} \end{aligned}$$

$$\begin{aligned} &= \max\{0, 0\} \\ &= 0. \end{aligned}$$

Therefore, $d(u, fu) = 0$, so $u = fu$. \square

Definition 5 Let (X, d) be a b -metric space. A mapping $f : X \rightarrow X$ is called a rational Geraghty contraction of type III if there exists $\beta \in \mathcal{F}$ such that

$$d(fx, fy) \leq \beta(M(x, y))M(x, y) \quad (2.16)$$

for all comparable elements $x, y \in X$, where

$$M(x, y) = \max \left\{ d(x, y), \frac{d(x, fx)d(y, fy)}{1 + s[d(x, y) + d(x, fy) + d(y, fx)]}, \frac{d(x, fy)d(x, y)}{1 + sd(x, fx) + s^3[d(y, fx) + d(y, fy)]} \right\}.$$

Theorem 7 Let (X, \leq) be a partially ordered set and suppose that there exists a b -metric d on X such that (X, d) is a b -complete b -metric space. Let $f : X \rightarrow X$ be an increasing mapping with respect to \leq such that there exists an element $x_0 \in X$ with $x_0 \leq f(x_0)$. Suppose f is a rational Geraghty contractive mapping of type III. If

- (I) f is continuous, or,
- (II) whenever $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \rightarrow u \in X$, one has $x_n \leq u$ for all $n \in \mathbb{N}$,

then f has a fixed point.

Moreover, the set of fixed points of f is well ordered if and only if f has one and only one fixed point.

Proof Set $x_n = f^n(x_0)$.

Step I: We show that $\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0$. Since $x_n \leq x_{n+1}$ for each $n \in \mathbb{N}$, then by (2.16)

$$\begin{aligned} d(x_n, x_{n+1}) &= d(fx_{n-1}, fx_n) \\ &\leq \beta(M(x_{n-1}, x_n))M(x_{n-1}, x_n) \\ &\leq \beta(d(x_{n-1}, x_n))d(x_{n-1}, x_n) \\ &< \frac{1}{s}d(x_{n-1}, x_n) \\ &\leq d(x_{n-1}, x_n), \end{aligned} \quad (2.17)$$

because

$$\begin{aligned} M(x_{n-1}, x_n) &= \max \left\{ d(x_{n-1}, x_n), \frac{d(x_{n-1}, fx_{n-1})d(x_n, fx_n)}{1 + s[d(x_{n-1}, x_n) + d(x_{n-1}, fx_n) + d(x_n, fx_{n-1})]}, \right. \\ &\quad \left. \frac{d(x_{n-1}, fx_n)d(x_{n-1}, x_n)}{1 + sd(x_{n-1}, fx_{n-1}) + s^3[d(x_n, fx_{n-1}) + d(x_n, fx_n)]} \right\} \\ &= \max \left\{ d(x_{n-1}, x_n), \frac{d(x_{n-1}, x_n)d(x_n, x_{n+1})}{1 + s[d(x_{n-1}, x_n) + d(x_{n-1}, x_{n+1}) + d(x_n, x_n)]} \right\}, \end{aligned}$$

$$\begin{aligned} & \left. \frac{d(x_{n-1}, x_{n+1})d(x_{n-1}, x_n)}{1 + sd(x_{n-1}, x_n) + s^3[d(x_n, x_n) + d(x_n, x_{n+1})]} \right\} \\ & \leq \max \left\{ d(x_{n-1}, x_n), \frac{d(x_{n-1}, x_n)s[d(x_n, x_{n-1}) + d(x_{n-1}, x_{n+1})]}{s[d(x_{n-1}, x_n) + d(x_{n-1}, x_{n+1}) + d(x_n, x_n)]} \right\} \\ & = d(x_{n-1}, x_n). \end{aligned}$$

Therefore, $\{d(x_n, x_{n+1})\}$ is decreasing. Similar to what we have done in Theorems 5 and 6, we have

$$\lim_{n \rightarrow \infty} d(x_{n-1}, x_n) = 0. \quad (2.18)$$

Step II: Now, we prove that the sequence $\{x_n\}$ is a b -Cauchy sequence. Suppose the contrary, *i.e.*, $\{x_n\}$ is not a b -Cauchy sequence. Then there exists $\varepsilon > 0$ for which we can find two subsequences $\{x_{m_i}\}$ and $\{x_{n_i}\}$ of $\{x_n\}$ such that n_i is the smallest index for which

$$n_i > m_i > i \quad \text{and} \quad d(x_{m_i}, x_{n_i}) \geq \varepsilon. \quad (2.19)$$

This means that

$$d(x_{m_i}, x_{n_i-1}) < \varepsilon. \quad (2.20)$$

From (2.18) and using the triangular inequality, we get

$$\varepsilon \leq d(x_{m_i}, x_{n_i}) \leq sd(x_{m_i}, x_{m_i+1}) + sd(x_{m_i+1}, x_{n_i}).$$

By taking the upper limit as $i \rightarrow \infty$, we get

$$\frac{\varepsilon}{s} \leq \limsup_{i \rightarrow \infty} d(x_{m_i+1}, x_{n_i}). \quad (2.21)$$

Using the triangular inequality, we have

$$d(x_{m_i}, x_{n_i}) \leq sd(x_{m_i}, x_{n_i-1}) + sd(x_{n_i-1}, x_{n_i}).$$

Taking the upper limit as $i \rightarrow \infty$ in the above inequality and using (2.20) we get

$$\limsup_{i \rightarrow \infty} d(x_{m_i}, x_{n_i}) \leq \varepsilon s. \quad (2.22)$$

Again, using the triangular inequality, we have

$$d(x_{m_i}, x_{n_i}) \leq sd(x_{m_i}, x_{m_i+1}) + s^2 d(x_{m_i+1}, x_{n_i-1}) + s^2 d(x_{n_i-1}, x_{n_i}).$$

Taking the upper limit as $i \rightarrow \infty$ in the above inequality and using (2.20) we get

$$\limsup_{i \rightarrow \infty} d(x_{m_i+1}, x_{n_i-1}) \geq \frac{\varepsilon}{s^2}. \quad (2.23)$$

From the definition of $M(x, y)$ and the above limits,

$$\begin{aligned} & \limsup_{i \rightarrow \infty} M(x_{m_i}, x_{n_i-1}) \\ &= \limsup_{i \rightarrow \infty} \max \left\{ d(x_{m_i}, x_{n_i-1}), \frac{d(x_{m_i}, fx_{m_i})d(x_{n_i-1}, fx_{n_i-1})}{1 + s[d(x_{m_i}, x_{n_i-1}) + d(x_{m_i}, fx_{n_i-1}) + d(x_{n_i-1}, fx_{m_i})]}, \right. \\ & \quad \left. \frac{d(x_{m_i}, fx_{n_i-1})d(x_{m_i}, x_{n_i-1})}{1 + sd(x_{m_i}, fx_{m_i}) + s^3[d(x_{n_i-1}, fx_{m_i}) + d(x_{n_i-1}, fx_{n_i-1})]} \right\} \\ &= \limsup_{i \rightarrow \infty} \max \left\{ d(x_{m_i}, x_{n_i-1}), \frac{d(x_{m_i}, x_{m_i+1})d(x_{n_i-1}, x_{n_i})}{1 + s[d(x_{m_i}, x_{n_i-1}) + d(x_{m_i}, x_{n_i}) + d(x_{n_i-1}, x_{m_i+1})]}, \right. \\ & \quad \left. \frac{d(x_{m_i}, x_{n_i})d(x_{m_i}, x_{n_i-1})}{1 + sd(x_{m_i}, x_{m_i+1}) + s^3[d(x_{n_i-1}, x_{m_i+1}) + d(x_{n_i-1}, x_{n_i})]} \right\} \\ &\leq \varepsilon. \end{aligned}$$

Now, from (2.16) and the above inequalities, we have

$$\begin{aligned} & \frac{\varepsilon}{s} \leq \limsup_{i \rightarrow \infty} d(x_{m_i+1}, x_{n_i}) \\ &\leq \limsup_{i \rightarrow \infty} \beta(M(x_{m_i}, x_{n_i-1})) \limsup_{i \rightarrow \infty} M(x_{m_i}, x_{n_i-1}) \\ &\leq \varepsilon \limsup_{i \rightarrow \infty} \beta(M(x_{m_i}, x_{n_i-1})), \end{aligned}$$

which implies that $\frac{1}{s} \leq \limsup_{i \rightarrow \infty} \beta(M(x_{m_i}, x_{n_i-1}))$. Now, as $\beta \in \mathcal{F}$ we conclude that $\{x_n\}$ is a b -Cauchy sequence. b -Completeness of X shows that $\{x_n\}$ b -converges to a point $u \in X$.

Step III: u is a fixed point of f .

When f is continuous, the proof is straightforward.

Now, let (II) hold. By Lemma 1

$$\begin{aligned} & \frac{1}{s} d(u, fu) \leq \limsup_{n \rightarrow \infty} d(x_{n+1}, fu) \\ &\leq \limsup_{n \rightarrow \infty} \beta(M(x_n, u)) \limsup_{n \rightarrow \infty} M(x_n, u), \end{aligned}$$

where

$$\begin{aligned} \lim_{n \rightarrow \infty} M(x_n, u) &= \lim_{n \rightarrow \infty} \max \left\{ d(x_n, u), \frac{d(x_n, fx_n)d(u, fu)}{1 + s[d(x_n, u) + d(x_n, fu) + d(u, fx_n)]}, \right. \\ & \quad \left. \frac{d(x_n, fu)d(x_n, u)}{1 + sd(x_n, fx_n) + s^3[d(u, fu) + d(u, fx_n)]} \right\} \\ &= \max\{0, 0\} \\ &= 0. \end{aligned}$$

Therefore, from the above relations, we deduce that $d(u, fu) = 0$, so $u = fu$. \square

If in the above theorems we take $\beta(t) = r$, where $0 \leq r < \frac{1}{s}$, then we have the following corollary.

Corollary 1 Let (X, \preceq) be a partially ordered set and suppose that there exists a b -metric d on X such that (X, d) is a b -complete b -metric space, and let $f : X \rightarrow X$ be an increasing mapping with respect to \preceq such that there exists an element $x_0 \in X$ with $x_0 \preceq f(x_0)$. Suppose that

$$d(fx, fy) \leq rM(x, y)$$

for all comparable elements $x, y \in X$, where

$$M(x, y) = \max \left\{ d(x, y), \frac{d(x, fx)d(y, fy)}{1 + d(x, y)}, \frac{d(x, fx)d(y, fy)}{1 + d(fx, fy)} \right\}$$

or

$$M(x, y) = \max \left\{ d(x, y), \frac{d(x, fx)d(x, fy) + d(y, fy)d(y, fx)}{1 + s[d(x, fx) + d(y, fy)]}, \frac{d(x, fx)d(x, fy) + d(y, fy)d(y, fx)}{1 + d(x, fy) + d(y, fx)} \right\},$$

or

$$M(x, y) = \max \left\{ d(x, y), \frac{d(x, fx)d(y, fy)}{1 + s[d(x, y) + d(x, fy) + d(y, fx)]}, \frac{d(x, fy)d(x, y)}{1 + sd(x, fx) + s^3[d(y, fx) + d(y, fy)]} \right\}.$$

If f is continuous, or, for any nondecreasing sequence $\{x_n\}$ in X such that $x_n \rightarrow u \in X$ one has $x_n \preceq u$ for all $n \in \mathbb{N}$, then f has a fixed point.

Corollary 2 Let (X, \preceq) be a partially ordered set and suppose that there exists a b -metric d on X such that (X, d) is a b -complete b -metric space, and let $f : X \rightarrow X$ be an increasing mapping with respect to \preceq such that there exists an element $x_0 \in X$ with $x_0 \preceq f(x_0)$. Suppose

$$d(fx, fy) \leq ad(x, y) + b \frac{d(x, fx)d(y, fy)}{1 + d(x, y)} + c \frac{d(x, fx)d(y, fy)}{1 + d(fx, fy)}$$

or

$$d(fx, fy) \leq ad(x, y) + b \frac{d(x, fx)d(x, fy) + d(y, fy)d(y, fx)}{1 + s[d(x, fx) + d(y, fy)]} + c \frac{d(x, fx)d(x, fy) + d(y, fy)d(y, fx)}{1 + d(x, fy) + d(y, fx)},$$

or

$$d(fx, fy) \leq ad(x, y) + b \frac{d(x, fx)d(y, fy)}{1 + s[d(x, y) + d(x, fy) + d(y, fx)]} + c \frac{d(x, fy)d(x, y)}{1 + sd(x, fx) + s^3[d(y, fx) + d(y, fy)]}$$

for all comparable elements $x, y \in X$, where $a, b, c \geq 0$ and $0 \leq a + b + c < \frac{1}{s}$.

If f is continuous, or, for any nondecreasing sequence $\{x_n\}$ in X such that $x_n \rightarrow u \in X$ one has $x_n \leq u$ for all $n \in \mathbb{N}$, then f has a fixed point.

Corollary 3 Let (X, \leq, d) be an ordered b -complete b -metric space, and let $f : X \rightarrow X$ be an increasing mapping with respect to \leq such that there exists an element $x_0 \in X$ with $x_0 \leq f^m(x_0)$ and

$$d(f^m x, f^m y) \leq \beta(M(x, y))M(x, y)$$

for all comparable elements $x, y \in X$, where

$$M(x, y) = \max \left\{ d(x, y), \frac{d(x, f^m x)d(y, f^m y)}{1 + d(x, y)}, \frac{d(x, f^m x)d(y, f^m y)}{1 + d(f^m x, f^m y)} \right\}$$

or

$$M(x, y) = \max \left\{ d(x, y), \frac{d(x, f^m x)d(x, f^m y) + d(y, f^m y)d(y, f^m x)}{1 + s[d(x, f^m x) + d(y, f^m y)]}, \frac{d(x, f^m x)d(x, f^m y) + d(y, f^m y)d(y, f^m x)}{1 + d(x, f^m y) + d(y, f^m x)} \right\},$$

or

$$M(x, y) = \max \left\{ d(x, y), \frac{d(x, f^m x)d(y, f^m y)}{1 + s[d(x, y) + d(x, f^m y) + d(y, f^m x)]}, \frac{d(x, f^m y)d(x, y)}{1 + sd(x, f^m x) + s^3[d(y, f^m x) + d(y, f^m y)]} \right\}$$

for some positive integer m .

If f^m is continuous, or, for any nondecreasing sequence $\{x_n\}$ in X such that $x_n \rightarrow u \in X$ one has $x_n \leq u$ for all $n \in \mathbb{N}$, then f has a fixed point.

Let Ψ be the family of all nondecreasing functions $\psi : [0, \infty) \rightarrow [0, \infty)$ such that

$$\lim_{n \rightarrow \infty} \psi^n(t) = 0$$

for all $t > 0$.

Lemma 2 If $\psi \in \Psi$, then the following are satisfied.

- (a) $\psi(t) < t$ for all $t > 0$;
- (b) $\psi(0) = 0$.

As an example $\psi_1(t) = kt$, for all $t \geq 0$, where $k \in [0, 1)$, and $\psi_2(t) = \ln(t + 1)$, for all $t \geq 0$, are in Ψ .

Theorem 8 Let (X, \leq) be a partially ordered set and suppose that there exists a b -metric d on X such that (X, d) is a b -complete b -metric space, and let $f : X \rightarrow X$ be an increasing mapping with respect to \leq such that there exists an element $x_0 \in X$ with $x_0 \leq f(x_0)$. Suppose that

$$sd(fx, fy) \leq \psi(M(x, y)), \quad (2.24)$$

where

$$M(x, y) = \max \left\{ d(x, y), \frac{d(x, fx)d(y, fy)}{1 + d(x, y)}, \frac{d(x, fx)d(y, fy)}{1 + d(fx, fy)} \right\}$$

for all comparable elements $x, y \in X$. If f is continuous, then f has a fixed point. Moreover, the set of fixed points of f is well ordered if and only if f has one and only one fixed point.

Proof Since $x_0 \leq f(x_0)$ and f is increasing, we obtain by induction that

$$x_0 \leq f(x_0) \leq f^2(x_0) \leq \cdots \leq f^n(x_0) \leq f^{n+1}(x_0) \leq \cdots$$

Putting $x_n = f^n(x_0)$, we have

$$x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_n \leq x_{n+1} \leq \cdots$$

If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$ then $x_{n_0} = fx_{n_0}$ and so we have nothing to prove. Hence, we assume that $d(x_n, x_{n+1}) > 0$, for all $n \in \mathbb{N}$.

In the following steps, we will complete the proof.

Step I: We will prove that

$$\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0.$$

Using condition (2.24), we obtain

$$d(x_{n+1}, x_n) \leq sd(x_{n+1}, x_n) = sd(fx_n, fx_{n-1}) \leq \psi(M(x_n, x_{n-1})),$$

because

$$\begin{aligned} M(x_{n-1}, x_n) &= \max \left\{ d(x_{n-1}, x_n), \frac{d(x_{n-1}, fx_{n-1})d(x_n, fx_n)}{1 + d(x_{n-1}, x_n)}, \right. \\ &\quad \left. \frac{d(x_{n-1}, fx_{n-1})d(x_n, fx_n)}{1 + d(fx_{n-1}, fx_n)} \right\} \\ &= \max \left\{ d(x_{n-1}, x_n), \frac{d(x_{n-1}, x_n)d(x_n, x_{n+1})}{1 + d(x_{n-1}, x_n)}, \frac{d(x_{n-1}, x_n)d(x_n, x_{n+1})}{1 + d(x_n, x_{n+1})} \right\} \\ &\leq \max \{ d(x_{n-1}, x_n), d(x_n, x_{n+1}) \}. \end{aligned}$$

If $\max \{ d(x_{n-1}, x_n), d(x_n, x_{n+1}) \} = d(x_n, x_{n+1})$, then

$$\begin{aligned} d(x_n, x_{n+1}) &\leq sd(x_n, x_{n+1}) = sd(fx_{n-1}, x_n) \\ &\leq \psi(M(x_{n-1}, x_n)) < M(x_{n-1}, x_n) \leq d(x_n, x_{n+1}), \end{aligned} \quad (2.25)$$

which is a contradiction. Hence, $\max \{ d(x_{n-1}, x_n), d(x_n, x_{n+1}) \} = d(x_{n-1}, x_n)$, so from (2.25),

$$\begin{aligned} d(x_n, x_{n+1}) &\leq sd(x_n, x_{n+1}) = sd(fx_{n-1}, x_n) \\ &\leq \psi(M(x_{n-1}, x_n)) < M(x_{n-1}, x_n) \leq d(x_{n-1}, x_n). \end{aligned} \quad (2.26)$$

Hence,

$$d(x_n, x_{n+1}) \leq sd(x_n, x_{n+1}) \leq \psi(d(x_{n-1}, x_n)).$$

By induction,

$$\begin{aligned} d(x_{n+1}, x_n) &\leq \psi(d(x_n, x_{n-1})) \leq \psi^2(d(x_{n-1}, x_{n-2})) \\ &\leq \dots \leq \psi^n(d(x_1, x_0)). \end{aligned} \quad (2.27)$$

As $\psi \in \Psi$, we conclude that

$$\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0. \quad (2.28)$$

Step II: Now, we prove that the sequence $\{x_n\}$ is a b -Cauchy sequence. Suppose the contrary, i.e., $\{x_n\}$ is not a b -Cauchy sequence. Then there exists $\varepsilon > 0$ for which we can find two subsequences $\{x_{m_i}\}$ and $\{x_{n_i}\}$ of $\{x_n\}$ such that n_i is the smallest index for which

$$n_i > m_i > i \quad \text{and} \quad d(x_{m_i}, x_{n_i}) \geq \varepsilon. \quad (2.29)$$

This means that

$$d(x_{m_i}, x_{n_i-1}) < \varepsilon. \quad (2.30)$$

From (2.29) and using the triangular inequality, we get

$$\varepsilon \leq d(x_{m_i}, x_{n_i}) \leq sd(x_{m_i}, x_{m_i+1}) + sd(x_{m_i+1}, x_{n_i}).$$

Taking the upper limit as $i \rightarrow \infty$, we get

$$\frac{\varepsilon}{s} \leq \limsup_{i \rightarrow \infty} d(x_{m_i+1}, x_{n_i}). \quad (2.31)$$

From the definition of $M(x, y)$ and the above limits,

$$\begin{aligned} &\limsup_{i \rightarrow \infty} M(x_{m_i}, x_{n_i-1}) \\ &= \limsup_{i \rightarrow \infty} \max \left\{ d(x_{m_i}, x_{n_i-1}), \frac{d(x_{m_i}, x_{m_i+1})d(x_{n_i-1}, x_{n_i})}{1 + d(x_{m_i}, x_{n_i-1})}, \right. \\ &\quad \left. \frac{d(x_{m_i}, x_{m_i+1})d(x_{n_i-1}, x_{n_i})}{1 + d(x_{m_i}, x_{n_i-1})} \right\} \\ &= \limsup_{i \rightarrow \infty} \max \left\{ d(x_{m_i}, x_{n_i-1}), \frac{d(x_{m_i}, x_{m_i+1})d(x_{n_i-1}, x_{n_i})}{1 + d(x_{m_i}, x_{n_i-1})}, \right. \\ &\quad \left. \frac{d(x_{m_i}, x_{m_i+1})d(x_{n_i-1}, x_{n_i})}{1 + d(x_{m_i}, x_{n_i-1})} \right\} \\ &\leq \varepsilon. \end{aligned}$$

Now, from (2.24) and the above inequalities, we have

$$\begin{aligned}\varepsilon &= s \cdot \frac{\varepsilon}{s} \leq s \limsup_{i \rightarrow \infty} d(x_{m_i+1}, x_{n_i}) \\ &\leq \limsup_{i \rightarrow \infty} \psi(M(x_{m_i}, x_{n_i-1})) \\ &\leq \psi(\varepsilon) < \varepsilon,\end{aligned}$$

which is a contradiction. Consequently, $\{x_n\}$ is a b -Cauchy sequence. b -Completeness of X shows that $\{x_n\}$ b -converges to a point $u \in X$.

Step III: Now we show that u is a fixed point of f ,

$$u = \lim_{n \rightarrow \infty} x_{n+1} = \lim_{n \rightarrow \infty} fx_n = fu,$$

as f is continuous. □

Theorem 9 *Under the same hypotheses as Theorem 8, without the continuity assumption of f , assume that whenever $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \rightarrow u \in X$, $x_n \leq u$ for all $n \in \mathbb{N}$. Then f has a fixed point.*

Proof By repeating the proof of Theorem 8, we construct an increasing sequence $\{x_n\}$ in X such that $x_n \rightarrow u \in X$. Using the assumption on X we have $x_n \leq u$. Now we show that $u = fu$. By (2.24) we have

$$d(fu, x_n) = d(fu, fx_{n-1}) \leq \psi(M(u, x_{n-1})), \quad (2.32)$$

where

$$\begin{aligned}M(u, x_{n-1}) &= \max \left\{ d(u, x_{n-1}), \frac{d(u, fu)d(x_{n-1}, fx_{n-1})}{1 + d(fu, fx_{n-1})}, \frac{d(u, fu)d(x_{n-1}, fx_{n-1})}{1 + d(u, x_{n-1})} \right\} \\ &= \max \left\{ d(u, x_{n-1}), \frac{d(u, fu)d(x_{n-1}, x_n)}{1 + d(fu, x_n)}, \frac{d(u, fu)d(x_{n-1}, x_n)}{1 + d(u, x_{n-1})} \right\}.\end{aligned}$$

Letting $n \rightarrow \infty$,

$$\limsup_{n \rightarrow \infty} M(u, x_{n-1}) = 0. \quad (2.33)$$

Again, taking the upper limit as $n \rightarrow \infty$ in (2.32) and using Lemma 1 and (2.33),

$$\begin{aligned}\frac{1}{s}d(fu, u) &\leq \limsup_{n \rightarrow \infty} d(fu, x_n) \\ &\leq \limsup_{n \rightarrow \infty} \psi(M(u, x_{n-1})) \\ &= 0.\end{aligned}$$

So we get $d(fu, u) = 0$, i.e., $fu = u$. □

Remark 1 In Theorems 8 and 9, we can replace $M(x, y)$ by the following:

$$M(x, y) = \max \left\{ d(x, y), \frac{d(x, fx)d(x, fy) + d(y, fy)d(y, fx)}{1 + s[d(x, fx) + d(y, fy)]}, \right. \\ \left. \frac{d(x, fx)d(x, fy) + d(y, fy)d(y, fx)}{1 + d(x, fy) + d(y, fx)} \right\}$$

or

$$M(x, y) = \max \left\{ d(x, y), \frac{d(x, fx)d(y, fy)}{1 + s[d(x, y) + d(x, fy) + d(y, fx)]}, \right. \\ \left. \frac{d(x, fy)d(x, y)}{1 + sd(x, fx) + s^3[d(y, fx) + d(y, fy)]} \right\}.$$

Example 2 Let $X = \{0, 1, 3\}$ and define the partial order \leq on X by

$$\leq := \{(0, 0), (1, 1), (3, 3), (0, 3), (3, 1), (0, 1)\}.$$

Consider the function $f : X \rightarrow X$ given as

$$f = \begin{pmatrix} 0 & 1 & 3 \\ 3 & 1 & 1 \end{pmatrix},$$

which is increasing with respect to \leq . Let $x_0 = 0$. Hence, $f(x_0) = 3$, so $x_0 \leq fx_0$. Define first the b -metric d on X by $d(0, 1) = 6$, $d(0, 3) = 9$, $d(1, 3) = \frac{1}{2}$, and $d(x, x) = 0$. Then (X, d) is a b -complete b -metric space with $s = \frac{18}{13}$. Let $\beta \in \mathcal{F}$ is given by

$$\beta(t) = \frac{13}{18} e^{-\frac{t}{9}}, \quad t \geq 0$$

and $\beta(0) \in [0, \frac{13}{18})$. Then

$$d(f0, f3) = d(3, 1) = \frac{1}{2} \leq \beta(M(0, 3))M(0, 3) = 9\beta(9).$$

This is because

$$M(0, 3) = \max \left\{ d(0, 3), \frac{d(0, f0)d(3, f3)}{1 + d(f0, f3)}, \frac{d(0, f0)d(3, f3)}{1 + d(0, 3)} \right\} \\ = \max \left\{ d(0, 3), \frac{d(0, 3)d(3, 1)}{1 + d(3, 1)}, \frac{d(0, 3)d(3, 1)}{1 + d(0, 3)} \right\} = 9.$$

Also,

$$d(f0, f1) = d(3, 1) = \frac{1}{2} \leq \beta(M(0, 1))M(0, 1) = 6\beta(6),$$

because

$$M(0, 1) = \max \left\{ d(0, 1), \frac{d(0, f0)d(1, f1)}{1 + d(f0, f1)}, \frac{d(0, f0)d(1, f1)}{1 + d(0, 1)} \right\} \\ = \max \left\{ d(0, 1), \frac{d(0, 3)d(1, 1)}{1 + d(3, 1)}, \frac{d(0, 3)d(1, 1)}{1 + d(0, 1)} \right\} = 6.$$

Also,

$$d(f1, f3) = d(1, 1) = 0 \leq \beta(M(1, 3))M(1, 3).$$

Hence, f satisfies all the assumptions of Theorem 5 and thus it has a fixed point (which is $u = 1$).

Example 3 Let $X = [0, 1]$ be equipped with the usual order and b -complete b -metric given by $d(x, y) = |x - y|^2$ with $s = 2$. Consider the mapping $f : X \rightarrow X$ defined by $f(x) = \frac{1}{16}x^2e^{-x^2}$ and the function β given by $\beta(t) = \frac{1}{4}$. It is easy to see that f is an increasing function and $0 \leq f(0) = 0$. For all comparable elements $x, y \in X$, by the mean value theorem, we have

$$\begin{aligned} d(fx, fy) &= \left| \frac{1}{16}x^2e^{-x^2} - \frac{1}{16}y^2e^{-y^2} \right|^2 \\ &\leq \frac{1}{8} |x^2e^{-x^2} - y^2e^{-y^2}|^2 \\ &\leq \frac{1}{8} |x - y|^2 \leq \frac{1}{4} d(x, y) = \beta(d(x, y))d(x, y) \\ &\leq \beta(M(x, y))M(x, y). \end{aligned}$$

So, from Theorem 5, f has a fixed point.

Example 4 Let $X = [0, 1]$ be equipped with the usual order and b -complete b -metric d be given by $d(x, y) = |x - y|^2$ with $s = 2$. Consider the mapping $f : X \rightarrow X$ defined by $f(x) = \frac{1}{4}\ln(x^2 + 1)$ and the function $\psi \in \Psi$ given by $\psi(t) = \frac{1}{4}t$, $t \geq 0$. It is easy to see that f is increasing and $0 \leq f(0) = 0$. For all comparable elements $x, y \in X$, using the mean value problem, we have

$$\begin{aligned} d(fx, fy) &= \left| \frac{1}{4}\ln(x^2 + 1) - \frac{1}{4}\ln(y^2 + 1) \right|^2 \\ &\leq \frac{1}{4} |x - y|^2 \\ &= \frac{1}{4} d(x, y) = \psi(d(x, y)) \leq \psi(M(x, y)), \end{aligned}$$

so, using Theorem 8, f has a fixed point.

3 Application

In this section, we present an application where Theorem 8 can be applied. This application is inspired by [9] (also, see [26] and [27]).

Let $X = C([0, T])$ be the set of all real continuous functions on $[0, T]$. We first endow X with the b -metric

$$d(u, v) = \max_{t \in [0, T]} (|u(t) - v(t)|)^p$$

for all $u, v \in X$ where $p > 1$. Clearly, (X, d) is a complete b -metric space with parameter $s = 2^{p-1}$. Secondly, $C([0, T])$ can also be equipped with a partial order given by

$$x \leq y \quad \text{iff} \quad x(t) \leq y(t) \quad \text{for all } t \in [0, T].$$

Moreover, as in [9] it is proved that $(C([0, T]), \leq)$ is regular, that is, whenever $\{x_n\}$ in X is an increasing sequence such that $x_n \rightarrow x$ as $n \rightarrow \infty$, we have $x_n \leq x$ for all $n \in \mathbb{N} \cup \{0\}$.

Consider the first-order periodic boundary value problem

$$\begin{cases} x'(t) = f(t, x(t)), \\ x(0) = x(T), \end{cases} \quad (3.1)$$

where $t \in I = [0, T]$ with $T > 0$ and $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function.

A lower solution for (3.1) is a function $\alpha \in C^1[0, T]$ such that

$$\begin{cases} \alpha'(t) \leq f(t, \alpha(t)), \\ \alpha(0) \leq \alpha(T), \end{cases} \quad (3.2)$$

where $t \in I = [0, T]$.

Assume that there exists $\lambda > 0$ such that for all $x, y \in X$ we have

$$|f(t, x(t)) + \lambda x(t) - f(t, y(t)) - \lambda y(t)| \leq \frac{\lambda}{2^{p-1}} \sqrt[p]{\ln(|x(t) - y(t)|^p + 1)}. \quad (3.3)$$

Then the existence of a lower solution for (3.1) provides the existence of a unique solution of (3.1).

Problem (3.1) can be rewritten as

$$\begin{cases} x'(t) + \lambda x(t) = f(t, x(t)) + \lambda x(t), \\ x(0) = x(T). \end{cases}$$

Consider

$$\begin{cases} x'(t) + \lambda x(t) = \delta(t) = F(t, x(t)), \\ x(0) = x(T), \end{cases}$$

where $t \in I$.

Using the variation of parameters formula, we get

$$x(t) = x(0)e^{-\lambda t} + \int_0^t e^{-\lambda(t-s)} \delta(s) ds, \quad (3.4)$$

which yields

$$x(T) = x(0)e^{-\lambda T} + \int_0^T e^{-\lambda(T-s)} \delta(s) ds.$$

Since $x(0) = x(T)$, we get

$$x(0)[1 - e^{-\lambda T}] = e^{-\lambda T} \int_0^T e^{\lambda s} \delta(s) ds$$

or

$$x(0) = \frac{1}{e^{\lambda T} - 1} \int_0^T e^{\lambda s} \delta(s) ds.$$

Substituting the value of $x(0)$ in (3.4) we arrive at

$$x(t) = \int_0^T G(t,s) \delta(s) ds,$$

where

$$G(t,s) = \begin{cases} \frac{e^{\lambda(T+s-t)}}{e^{\lambda T}-1}, & 0 \leq s \leq t \leq T, \\ \frac{e^{\lambda(s-t)}}{e^{\lambda T}-1}, & 0 \leq t \leq s \leq T. \end{cases}$$

Now define the operator $S : C[0, T] \rightarrow C[0, T]$ by

$$Sx(t) = \int_0^T G(t,s) F(s, x(s)) ds.$$

The mapping S is nondecreasing [26]. Note that if $u \in C[0, T]$ is a fixed point of S then $u \in C^1[0, T]$ is a solution of (3.1).

Let $x, y \in X$. Then we have

$$\begin{aligned} 2^{p-1} |Sx(t) - Sy(t)| &= 2^{p-1} \left| \int_0^T G(t,s) F(s, x(s)) ds - \int_0^T G(t,s) F(s, y(s)) ds \right| \\ &\leq 2^{p-1} \int_0^T |G(t,s)| [|F(s, x(s)) - F(s, y(s))|] ds \\ &\leq 2^{p-1} \int_0^T |G(t,s)| \frac{\lambda}{2^{p-1}} \sqrt[p]{\ln(|x(t) - y(t)|^p + 1)} ds \\ &\leq \lambda \sqrt[p]{\ln(d(x, y) + 1)} \left[\int_0^t \frac{e^{\lambda(T+s-t)}}{e^{\lambda T}-1} ds + \int_t^T \frac{e^{\lambda(s-t)}}{e^{\lambda T}-1} ds \right] \\ &= \lambda \sqrt[p]{\ln(d(x, y) + 1)} \left[\frac{1}{\lambda(e^{\lambda T}-1)} (e^{\lambda(T+s-t)}|_0^t + e^{\lambda(s-t)}|_t^T) \right] \\ &= \lambda \sqrt[p]{\ln(d(x, y) + 1)} \left[\frac{1}{\lambda(e^{\lambda T}-1)} (e^{\lambda T} - e^{\lambda(T-t)} + e^{\lambda(T-t)} - 1) \right] \\ &= \sqrt[p]{\ln(d(x, y) + 1)} \\ &\leq \sqrt[p]{\ln(M(x, y) + 1)}, \end{aligned}$$

or, equivalently,

$$2^{p-1} (|Sx(t) - Sy(t)|)^p \leq \ln(M(x, y) + 1),$$

which shows that

$$2^{p-1} d(Sx, Sy) \leq \ln(M(x, y) + 1),$$

where

$$M(x, y) = \max \left\{ d(x, y), \frac{d(x, Sx)d(y, Sy)}{1 + d(x, y)}, \frac{d(x, Sx)d(y, Sy)}{1 + d(Sx, Sy)} \right\}$$

or

$$M(x, y) = \max \left\{ d(x, y), \frac{d(x, Sx)d(x, Sy) + d(y, Sy)d(y, Sx)}{1 + 2^{p-1}[d(x, Sx) + d(y, Sy)]}, \right. \\ \left. \frac{d(x, Sx)d(x, Sy) + d(y, Sy)d(y, Sx)}{1 + d(x, Sy) + d(y, Sx)} \right\},$$

or

$$M(x, y) = \max \left\{ d(x, y), \frac{d(x, Sx)d(y, Sy)}{1 + 2^{p-1}[d(x, y) + d(x, Sy) + d(y, Sx)]}, \right. \\ \left. \frac{d(x, Sy)d(x, y)}{1 + 2^{p-1}d(x, Sx) + 2^{3p-3}[d(y, Sx) + d(y, Sy)]} \right\}.$$

Finally, let α be a lower solution for (3.1). In [26] it was shown that $\alpha \leq S(\alpha)$.

Hence, the hypotheses of Theorem 8 are satisfied with $\psi(t) = \ln(t + 1)$. Therefore, there exists a fixed point $\hat{x} \in C[0, T]$ such that $S\hat{x} = \hat{x}$.

Remark 2 In the above theorem, we can replace (3.3) by the following inequality:

$$|f(t, x(t)) + \lambda x(t) - f(t, y(t)) - \lambda y(t)| \leq \frac{\lambda}{2^{\frac{p^2-1}{p}}} \sqrt[p]{e^{-M(x,y)} M(x, y)} \quad (3.5)$$

for all $x \neq y \in X$.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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